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**An inequality of rearrangements
on the unit circle**

by

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AN INEQUALITY OF REARRANGEMENTS ON THE UNIT CIRCLE

CRISTINA DRAGHICI

ABSTRACT. We prove that the integral of the product of two functions over a symmetric set in $\mathbb{S}^1 \times \mathbb{S}^1$, defined as $E = \{(x, y) \in \mathbb{S}^1 \times \mathbb{S}^1 : d(\sigma_1(x), \sigma_2(y)) \leq \alpha\}$, where σ_1, σ_2 are diffeomorphisms of \mathbb{S}^1 with certain properties and d is the geodesic distance on \mathbb{S}^1 , increases when we pass to their symmetric decreasing rearrangement. We also give a characterization of these diffeomorphisms σ_1, σ_2 for which the rearrangement inequality holds. As a consequence, we obtain the result for the integral of the function $\Psi(f(x), g(y))$ (Ψ a supermodular function) with a kernel given as $k[d(\sigma_1(x), \sigma_2(y))]$, with k decreasing.

1. INTRODUCTION

On a measure space (X, μ) , the Hardy-Littlewood inequality asserts [4]:

$$\int_X f(x)g(x) d\mu(x) \leq \int_0^{\mu(X)} f^*(t)g^*(t) dt,$$

where f^* and g^* are the decreasing rearrangements of f and g , respectively. In what follows, $X = \mathbb{S}^1$, or $X = [-\pi, \pi]$, and the above inequality can be written as:

$$(1.1) \quad \int_{-\pi}^{\pi} f(x)g(x) dx \leq \int_{-\pi}^{\pi} f^\sharp(x)g^\sharp(x) dx,$$

with f^\sharp, g^\sharp the symmetric decreasing rearrangements of f and g , given by $f^\sharp(x) = f^*(2|x|)$ and $g^\sharp(x) = g^*(2|x|)$.

These inequalities can be proved using *the layer-cake formula* [10]: Every measurable function $f : X \rightarrow \mathbb{R}_+$ can be written as an integral of the characteristic function of its level sets:

$$(1.2) \quad f(x) = \int_0^\infty \chi_{\{f>t\}}(x) dt.$$

A more general rearrangement inequality on $X = \mathbb{R}^n$ is the Riesz-Sobolev inequality:

$$(1.3) \quad \int_{\mathbb{R}^{2n}} f(x)g(y)h(x-y) dx dy \leq \int_{\mathbb{R}^{2n}} f^\sharp(x)g^\sharp(x)h^\sharp(x-y) dx dy,$$

where f, g, h are non-negative functions which vanish at infinity in a weak sense. The case $n = 1$ is due to Riesz in 1930 (see [12]), and the case $n > 1$ is due to Sobolev in 1938 (see [13]). The proof can be found in the book by Hardy, Littlewood, Pólya [9] which sets the beginning of the systematic study of rearrangement inequalities. A more general version of this inequality in \mathbb{R}^n , involving n functions can be found in [5].

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The equivalent of (1.3) for three non-negative functions on the unit circle was proved by Baernstein [1]:

$$(1.4) \quad \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(e^{i\phi})g(e^{i\theta})h(e^{i(\phi-\theta)}) d\theta d\phi \leq \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f^{\sharp}(e^{i\phi})g^{\sharp}(e^{i\theta})h^{\sharp}(e^{i(\phi-\theta)}) d\theta d\phi.$$

The proof of this inequality uses a variational principle applied to the convolution of characteristic functions of sets which does not seem to generalize in higher dimensions.

The Riesz-Sobolev inequality (1.3) is equivalent to the Brunn-Minkowski inequality from convex geometry [8, 11, 7] which states that if K and L are measurable sets in \mathbb{R}^n , then their Minkowski (pointwise) sum $K + L$ is related to the measure of the sets K and L by

$$V(K + L)^{1/n} \geq V(K)^{1/n} + V(L)^{1/n},$$

where V denotes the n -dimensional volume. An analog of this inequality for \mathbb{S}^n is not known, and, since the proof of rearrangement inequalities in \mathbb{R}^n require it, an analog of the Riesz-Sobolev inequality (1.3) is not known in \mathbb{S}^n , for $n > 1$.

However, a partial result in \mathbb{S}^n was proved by Baernstein and Taylor in [2]. They considered a version of the Riesz-Sobolev inequality where one of the functions is symmetric decreasing. They showed that, if $h = K$ is already symmetric decreasing then

$$\int_{\mathbb{S}^n} \int_{\mathbb{S}^n} f(x)g(y)K(x \cdot y) d\sigma(x)d\sigma(y) \leq \int_{\mathbb{S}^n} \int_{\mathbb{S}^n} f^{\sharp}(x)g^{\sharp}(y)K(x \cdot y) d\sigma(x)d\sigma(y),$$

where $d\sigma$ is the surface measure on the unit sphere \mathbb{S}^n in \mathbb{R}^{n+1} , $x \cdot y$ is the usual inner product and $K(t)$ is an increasing function on $[-1, 1]$. Since $x \cdot y = \cos \alpha$, where α is the angle between the vectors x and y , we can write $K(x \cdot y) = k(d(x, y))$, with k decreasing. Here $d(x, y)$ is the great circle (geodesic) distance between x and y . Their proof is based on the polarization technique. They showed first that the inequality holds for the polarizations of f and g in any hyperplane and then they passed to the limit for the general case. They were led to this version of the Riesz-Sobolev inequality while trying to generalize a 2-dimensional result stating that u is subharmonic implies its star function is also subharmonic.

In this paper we are interested in the case $n = 1$ of this inequality with K replaced by the characteristic function of a symmetric set which does not depend on the distance between two points, but rather on the distance between their images under two diffeomorphisms σ_1, σ_2 of \mathbb{S}^1 . We will also obtain a characterization of these diffeomorphisms for which the inequality holds. With the set E defined as

$$E = \{(x, y) : d(\sigma_1(x), \sigma_2(y)) \leq \alpha\},$$

we will show that

$$(1.5) \quad \int_E f(x)g(y) dx dy \leq \int_E f^{\sharp}(x)g^{\sharp}(y) dx dy,$$

for every $\alpha > 0$. This result implies *the main result* of this paper, Theorem 3.6:

$$\begin{aligned} \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \Psi(f(x), g(y))k[d(\sigma_1(x), \sigma_2(y))] dx dy \\ \leq \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \Psi(f^{\sharp}(x), g^{\sharp}(y))k[d(\sigma_1(x), \sigma_2(y))] dx dy, \end{aligned}$$

with k decreasing and Ψ the distribution function of a measure μ .

The paper is organized as follows: We will first prove (1.5) for f and g replaced by characteristic functions χ_A , χ_B , and σ_2 the identity. Then we will deduce the result (1.5) mentioned above, and we will show that we can replace the product $f(x)g(y)$ by a function $\Psi(f(x), g(y))$ and that we can replace χ_E by a decreasing function of the distance between $\sigma_1(x)$ and $\sigma_2(y)$, yielding Theorem 3.6.

2. PRELIMINARIES

Recall that a function $f : I \rightarrow \mathbb{R}$, defined on an interval $I \subset \mathbb{R}$, is called convex if, for every $0 < \lambda < 1$ and every $a, b \in I$, the following inequality holds:

$$f(\lambda a + (1 - \lambda)b) \leq \lambda f(a) + (1 - \lambda)f(b).$$

A convex function is differentiable almost everywhere on I and its derivative is increasing.

We denote by \mathbb{S}^1 the unit circle in \mathbb{R}^2 , i.e., $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$, and by \mathbb{S}_+^1 the upper half unit circle,

$$\mathbb{S}_+^1 = \{e^{i\theta} : 0 \leq \theta \leq \pi\}.$$

Definition 2.1. A function $\sigma : \mathbb{S}_+^1 \rightarrow \mathbb{S}_+^1$ is called convex if the function $\sigma_1 : [0, \pi] \rightarrow [0, \pi]$, defined as :

$$\sigma(e^{i\theta}) = e^{i\sigma_1(\theta)}, \quad 0 \leq \theta \leq \pi,$$

is convex on $[0, \pi]$.

Let $f : \mathbb{S}^1 \rightarrow \mathbb{R}_+$ be a non-negative measurable function. We define its *distribution function*:

$$\lambda_f(t) = |\{f > t\}|, \quad t \in [0, \infty),$$

where $\{f > t\} := \{z \in \mathbb{S}^1 : f(z) > t\}$ denote the level sets of f , and $|A|$ is the linear measure on \mathbb{S}^1 of A . Functions which have the same distribution function are called *equimeasurable*.

We define the *symmetric decreasing rearrangement* of f to be the function $f^\sharp : \mathbb{S}^1 \rightarrow \mathbb{R}_+$, given by:

$$f^\sharp(z) = \inf\{t : \lambda_f(t) \leq 2d(1, z)\},$$

where $d(1, z)$ is the geodesic distance on \mathbb{S}^1 between z and 1.

It is clear that $f^\sharp(z) = f^\sharp(\bar{z})$ and that f^\sharp decreases as $d(1, z)$ increases. Also, f and f^\sharp are equimeasurable.

If we write $z = e^{i\theta}$, $-\pi \leq \theta < \pi$, then $d(1, z) = d(1, e^{i\theta}) = |\theta|$, and we can think of f as a function of θ via the relation

$$\tilde{f}(\theta) = f(e^{i\theta}).$$

For $\tilde{f} : [-\pi, \pi] \rightarrow \mathbb{R}_+$, one defines its symmetric decreasing rearrangement as:

$$\tilde{f}^\sharp(\theta) = \inf\{t : \lambda_{\tilde{f}}(t) \leq 2|\theta|\},$$

where, as before, $\lambda_{\tilde{f}}(t) = |\{\tilde{f} > t\}|$, and thus, there is a one-to-one correspondence between f^\sharp and \tilde{f}^\sharp , given by

$$\tilde{f}^\sharp(\theta) = f^\sharp(e^{i\theta}).$$

Whenever necessary, we will think of a function f defined on \mathbb{S}^1 as a function on $[-\pi, \pi]$. If $f = \chi_A$ is the characteristic function of a measurable set $A \subset \mathbb{S}^1$, then

$f^\sharp = \chi_{A^\sharp}$, where A^\sharp is the open interval on the unit circle centered at 1, having the same linear measure as A .

Next, we introduce the Hardy-Littlewood-Pólya preorder relation \prec for non-negative functions defined on the interval $[-\pi, \pi]$. We say that (see [3, 4]):

$$f \prec F \quad \text{iff} \quad \int_{-t}^t f^\sharp(s) ds \leq \int_{-t}^t F^\sharp(s) ds, \quad \text{for all } 0 \leq t \leq \pi.$$

This is equivalent to

$$\int_{-\pi}^{\pi} f^\sharp(s) h^\sharp(s) ds \leq \int_{-\pi}^{\pi} F^\sharp(s) h^\sharp(s) ds,$$

for every positive symmetric decreasing function h^\sharp defined on $[-\pi, \pi]$. To see this, write $h^\sharp(s) = \int_0^\infty \chi_{\{h^\sharp > t\}}(s) dt$ (this is the layer cake formula (1.2)), and, using Fubini's formula and the fact that $\{h^\sharp > t\} = (-l(t), l(t))$ is a symmetric interval,

$$\begin{aligned} \int_{-\pi}^{\pi} f^\sharp(s) h^\sharp(s) ds &= \int_0^\infty \left[\int_{-l(t)}^{l(t)} f^\sharp(s) ds \right] dt \\ &\leq \int_0^\infty \left[\int_{-l(t)}^{l(t)} F^\sharp(s) ds \right] dt = \int_{-\pi}^{\pi} F^\sharp(s) h^\sharp(s) ds. \end{aligned}$$

Yet another equivalent characterization is:

$$f \prec F \Leftrightarrow \int_E f(s) ds \leq \int_E F(s) ds, \quad \text{for every } E \subset [-\pi, \pi].$$

The next result is well-known and it follows from the proof of the equality case in the Hardy-Littlewood inequality, presented by Lieb and Loss in [10, pp.82]. We will include a proof here for consistency.

Lemma 2.2. *Let $f : [-\pi, \pi] \rightarrow \mathbb{R}_+$ be a measurable function such that*

$$(2.1) \quad \int_{-t}^t f(x) dx \geq \int_{-t}^t f^\sharp(x) dx, \quad \text{for every } 0 \leq t \leq \pi.$$

Then $f = f^\sharp$ a.e. on $[-\pi, \pi]$.

Proof. From (1.1) applied to $\chi_{(-t,t)}$ and f , it follows that we must have equality in (2.1), i.e.,

$$(2.2) \quad \int_{-t}^t f(x) dx = \int_{-t}^t f^\sharp(x) dx.$$

We will use the layer-cake formula to write $f(x) = \int_0^\infty \chi_{\{f > s\}}(x) ds$, and similarly for $f^\sharp(x)$.

Using (1.1), we obtain:

$$(2.3) \quad \int_{-t}^t \chi_{\{f > s\}}(x) dx \leq \int_{-t}^t \chi_{\{f^\sharp > s\}}(x) dx, \quad \text{for every } s \geq 0.$$

Fubini's theorem and (2.2) imply that:

$$\begin{aligned} \int_{-t}^t f(x) dx &= \int_0^\infty \left[\int_{-t}^t \chi_{\{f > s\}}(x) dx \right] ds \\ &= \int_0^\infty \left[\int_{-t}^t \chi_{\{f^\sharp > s\}}(x) dx \right] ds = \int_{-t}^t f^\sharp(x) dx. \end{aligned}$$

From this equality and (2.3) it follows that, for a fixed t , there exists a set of measure zero S_t , such that

$$\int_{-t}^t \chi_{\{f>s\}}(x) dx = \int_{-t}^t \chi_{\{f^\#>s\}}(x) dx, \quad \text{for every } s \in (0, \infty) \setminus S_t.$$

Next, we choose T_N a countable dense set in $[0, \pi]$ and we denote by $S_{T_N} = \cup_{t \in T_N} S_t$. Then:

$$(2.4) \quad \int_{-t}^t \chi_{\{f>s\}}(x) dx = \int_{-t}^t \chi_{\{f^\#>s\}}(x) dx, \quad \text{for every } t \in T_N \text{ and } s \in (0, \infty) \setminus S_{T_N}.$$

Since for every fixed s , $t \rightarrow \int_{-t}^t \chi_{\{f>s\}}(x) dx$ is a continuous function of t , in fact (2.4) holds for every $0 \leq t \leq \pi$. Thus,

$$\int_{-t}^t \chi_{\{f>s\}}(x) dx = \int_{-t}^t \chi_{\{f^\#>s\}}(x) dx, \quad \text{for all } 0 \leq t \leq \pi \text{ and a.e. } s \in (0, \infty).$$

Now, let t be such that $\{f^\# > s\} = (-t, t)$. Then, it follows that $\{f > s\} = (-t, t) = \{f^\# > s\}$ a.e., and thus, $f = f^\#$ by the layer cake formula. \square

The following result shows that $\int_{-t}^t f^\#(x) dx$ is attained as a supremum. A proof can be found in [4, Theorem 7.5, pp.82].

Theorem 2.3. (*J. V. Ryff*) *For every measurable function f as in Lemma 2.2, there exists a measure preserving transformation T such that $f = f^\# \circ T$. This guarantees, for every t , the existence of a set $A \subset [-\pi, \pi]$ of measure $2t$ such that $\int_A f(x) dx = \int_{-t}^t f^\#(x) dx$.*

3. MAIN RESULTS: INEQUALITIES ON THE CIRCLE

Notation. As before, d is the geodesic distance, also called the arclength, on the unit circle \mathbb{S}^1 . We have:

$$(3.1) \quad d(u, v) = d(u\bar{v}, 1), \quad \text{for all } u, v \in \mathbb{S}^1,$$

where \bar{v} denotes the complex conjugate of v .

We define, for $\alpha > 0$, the function:

$$\chi_\alpha(u, v) = \begin{cases} 1, & \text{if } d(u, v) \leq \alpha, \\ 0, & \text{otherwise} \end{cases}$$

and we observe that $\chi_\alpha(u, v) = \chi_\alpha(u\bar{v}, 1)$, by (3.1).

We introduce a new function, which we call again $\chi_\alpha : \mathbb{S}^1 \rightarrow \mathbb{R}_+$, given by $\chi_\alpha(z) := \chi_\alpha(z, 1)$, which is the characteristic function of the closed interval on \mathbb{S}^1 of linear length 2α , centered at 1.

We will make use, in what follows, of the relation:

$$(3.2) \quad \chi_\alpha(u\bar{v}) = \chi_\alpha(u, v), \quad \text{for all } u, v \in \mathbb{S}^1.$$

Given two positive measurable functions $f, g : \mathbb{S}^1 \rightarrow \mathbb{R}_+$, their convolution, $f * g$, is defined to be the function:

$$\begin{aligned} (f * g)(z_0) &= \int_{\mathbb{S}^1} f(z_0 \bar{z}) g(z) dz \\ &= \int_{-\pi}^{\pi} f(e^{i(\theta_0 - \theta)}) g(e^{i\theta}) d\theta, \end{aligned}$$

with $z_0 = e^{i\theta_0}$ and dz represents the arclength element on \mathbb{S}^1 , usually denoted by $|dz|$.

Given three positive functions f, g, h defined on \mathbb{S}^1 , we can write

$$(3.3) \quad \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(e^{i(\theta-t)})g(e^{it})h(e^{i\theta}) dt d\theta = (f * g * h^-)(1),$$

where $h^-(z) = h(\bar{z})$, i.e., $h^-(e^{i\theta}) = h(e^{-i\theta})$.

Theorem 3.1. *Let $\sigma : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be a C^1 diffeomorphism such that $\sigma(1) = 1$ and $\sigma(-1) = -1$. Additionally, we assume that $\sigma(\mathbb{S}_+^1) \subseteq \mathbb{S}_+^1$ and $\sigma(\mathbb{S}_-^1) \subseteq \mathbb{S}_-^1$. Let d be the geodesic distance on the unit circle, α be a positive real number, and we define the set $E = \{(x, y) \in \mathbb{S}^1 \times \mathbb{S}^1 : d(\sigma(x), y) \leq \alpha\}$. For $A, B \subset \mathbb{S}^1$ measurable sets, let*

$$I_\alpha(A, B) = \int_{\mathbb{S}^1 \times \mathbb{S}^1} \chi_A(x) \chi_B(y) \chi_E(x, y) dx dy.$$

Then, for any A, B measurable subsets of \mathbb{S}^1 , and $\alpha > 0$,

$$(3.4) \quad I_\alpha(A, B) \leq I_\alpha(A^\sharp, B^\sharp),$$

if and only if, σ is symmetric (i.e. $\overline{\sigma(z)} = \sigma(\bar{z})$, for every $z \in \mathbb{S}^1$) and convex on \mathbb{S}_+^1 .

Proof. Sufficiency. We define $\sigma_1 : [-\pi, \pi) \rightarrow [-\pi, \pi)$ by $e^{\sigma_1(\theta)} := \sigma(e^{i\theta})$ and we assume that σ_1 is convex on $(0, \pi)$. Using change of variables, $(\sigma(x), y) = (u, v)$, the integral I_α becomes:

$$I_\alpha(A, B) = \int_{\mathbb{S}^1 \times \mathbb{S}^1} \chi_{\sigma(A)}(u) \chi_B(v) \chi_\alpha(u, v) (\sigma^{-1})'(u) du dv.$$

With $\chi_\alpha(u, v) = \chi_\alpha(u\bar{v})$, as in (3.2), the above expression becomes:

$$(3.5) \quad I_\alpha(A, B) = \int_{\mathbb{S}^1 \times \mathbb{S}^1} \chi_{\sigma(A)}(u) \chi_B(v) \chi_\alpha(u\bar{v}) \psi(u) du dv,$$

where $\psi(e^{i\theta}) = \tau_1'(\theta)$ and τ_1 is defined by $\sigma^{-1}(e^{i\theta}) = e^{i\tau_1(\theta)}$, and is the inverse of σ_1 .

Thus, we can write using convolution and (3.3):

$$I_\alpha(A, B) = [(\chi_{\sigma(A)} \cdot \psi) * \chi_\alpha * \chi_B^-](1),$$

where we used the fact that χ_α is a symmetric function.

It was proved in [1] (see also (1.4)) by Baernstein that, for any three positive measurable functions f, g, h on \mathbb{S}^1 , the following inequality holds:

$$(3.6) \quad (f * g * h^-)(1) \leq (f^\sharp * g^\sharp * h^\sharp)(1).$$

One can replace h^- in the inequality above by h since they are equimeasurable functions. Thus, based on (3.6) and the fact that χ_α is symmetric decreasing, we conclude that:

$$(3.7) \quad I_\alpha(A, B) \leq [(\chi_{\sigma(A)} \cdot \psi)^\sharp * \chi_\alpha * \chi_{B^\sharp}](1).$$

Fact: If F is a positive symmetric decreasing function and if $f \prec F$ in the sense of Hardy-Littlewood-Pólya (i.e. $\sup_{|G|=2\theta} \int_G f \leq \int_{-\theta}^\theta F$), then f^\sharp in inequality (3.6) can be replaced by F . Indeed, $f \prec F$ is equivalent to $\int_{\mathbb{S}^1} f^\sharp(z) g^\sharp(z) dz \leq \int_{\mathbb{S}^1} F(z) g^\sharp(z) dz$,

for all positive symmetric decreasing functions g^\sharp . Now, since $g^\sharp * h^\sharp$ is symmetric decreasing and since the convolution $(f^\sharp * g^\sharp * h^\sharp)(1)$ can be written as the integral of the product $f^\sharp(z)(g^\sharp * h^\sharp)(z)$, we conclude that:

$$(f^\sharp * g^\sharp * h^\sharp)(1) \leq (F * g^\sharp * h^\sharp)(1).$$

Therefore, using (3.7) and the Fact, we can prove (3.4) if we show that $\chi_{\sigma(A)}\psi \prec \chi_{\sigma(A^\sharp)}\psi$, i.e.

$$(3.8) \quad \int_E \chi_{\sigma(A)}\psi \leq \int_{E^\sharp} \chi_{\sigma(A^\sharp)}\psi.$$

Let $E' = \sigma^{-1}(E)$, and $E'' = \sigma^{-1}(E^\sharp)$. With these notations, inequality (3.8) becomes:

$$\int_{A \cap E'} dx \leq \int_{A^\sharp \cap E''} dx,$$

or equivalently, $|A \cap E'| \leq |A^\sharp \cap E''|$, which is true if $|E'| \leq |E''|$, since E'' is symmetric. Since ψ is symmetric decreasing, we have that $\int_E \psi(u)du \leq \int_{E^\sharp} \psi(u)du$, which is equivalent to $\int_{\sigma^{-1}(E)} dx \leq \int_{\sigma^{-1}(E^\sharp)} dx$, using change of variables. The latter inequality simply states that $|E'| \leq |E''|$, and the proof of the sufficiency is now complete.

Necessity. Dividing (3.5) by 2α , and letting α tend to zero, we obtain:

$$I_0(A, B) = \int_{\mathbb{S}^1} \chi_{\sigma(A)}(u) \chi_B(u) \psi(u) du,$$

and inequality (3.4) implies that:

$$(3.9) \quad I_0(A, B) \leq I_0(A^\sharp, B^\sharp).$$

With the notation $\tau = \sigma^{-1}$, ψ the Jacobian of τ , and $x = \tau(u)$, I_0 becomes:

$$(3.10) \quad I_0(A, B) = \int_{\mathbb{S}^1} \chi_A(x) \chi_{\tau(B)}(x) dx = |A \cap \tau(B)|.$$

First, we will show that the symmetry condition is necessary. Suppose τ is not symmetric. Then, there exists a point $x = e^{i\theta}$ in \mathbb{S}_+^1 such that $\tau(x) \neq \tau(\bar{x})$. If we consider $A = \tau(\{e^{it} : |t| < \theta\})$ and $B = \{e^{it} : |t| < \theta\}$, then we have: $|A \cap \tau(B)| = |\tau(B)| > |A^\sharp \cap \tau(B^\sharp)|$, since $\tau(B^\sharp)$ is not symmetric and $|A| = |\tau(B)|$. But this contradicts (3.9) and therefore (3.4).

Suppose now that τ_1 is symmetric, but not concave (or, equivalently, σ_1 is symmetric, but σ_1 is not convex on $(0, \pi)$). Then, there exist $e^{ib}, e^{ic} \in \mathbb{S}_+^1$ with $b, c \in (0, \pi)$ such that:

$$(3.11) \quad \frac{\tau_1(b) + \tau_1(c)}{2} > \tau_1\left(\frac{b+c}{2}\right).$$

Without loss of generality we can assume that $b > c$ and let us denote by $a = \frac{b+c}{2}$. Letting $B = \{e^{it} : -c < t < b\}$, it follows that $B^\sharp = \{e^{it} : -a < t < a\}$. We calculate $|\tau(B)| = \tau_1(b) - \tau_1(-c) = \tau_1(b) + \tau_1(c)$ and $|\tau(B^\sharp)| = 2\tau_1(a)$.

From (3.11) we obtain that $|\tau(B)| > |\tau(B^\sharp)|$ which shows that $I_0(\mathbb{S}^1, B) > I_0(\mathbb{S}^1, B^\sharp)$ and contradicts (3.4). Therefore, τ must also be concave. \square

Theorem 3.2. *Suppose we have two functions σ_1, σ_2 satisfying the conditions of σ in Theorem 3.1 and define $E = \{(x, y) \in \mathbb{S}^1 \times \mathbb{S}^1 : d(\sigma_1(x), \sigma_2(y)) \leq \alpha\}$, for $\alpha \in \mathbb{R}_+$. Let*

$$I_\alpha(A, B) = \int_{\mathbb{S}^1 \times \mathbb{S}^1} \chi_A(x) \chi_B(y) \chi_E(x, y) dx dy.$$

Then, for any A, B subsets of \mathbb{S}^1 and $\alpha > 0$,

$$(3.12) \quad I_\alpha(A, B) \leq I_\alpha(A^\sharp, B^\sharp),$$

if and only if σ_1, σ_2 are symmetric and convex on \mathbb{S}_+^1 .

Proof. Sufficiency. Very similar to Theorem 3.1. Using change of variables, $(\sigma_1(x), \sigma_2(y)) = (u, v)$, the integral becomes:

$$I_\alpha(A, B) = \int_{\mathbb{S}^1 \times \mathbb{S}^1} \chi_{\sigma_1(A)}(u) \chi_{\sigma_2(B)}(v) \chi_\alpha(u\bar{v}) \psi_1(u) \psi_2(v) du dv,$$

where ψ_1, ψ_2 are defined similarly to ψ in Theorem 3.1 (see (3.5)). Using convolution, this integral can be written as:

$$I_\alpha(A, B) = [(\chi_{\sigma_1(A)} \cdot \psi_1) * \chi_\alpha * (\chi_{\sigma_2(B)} \cdot \psi_2)^-](1).$$

We have already proven that $\chi_{\sigma_1(A)} \psi_1 \prec \chi_{\sigma_1(A^\sharp)} \psi_1$ and $\chi_{\sigma_2(B)} \psi_2 \prec \chi_{\sigma_2(B^\sharp)} \psi_2$, from which it follows that $I_\alpha(A, B) \leq I_\alpha(A^\sharp, B^\sharp)$.

Necessity. Using change of variable $v = \sigma_2(y)$, I_α becomes:

$$I_\alpha(A, B) = \int_{\mathbb{S}^1 \times \mathbb{S}^1} \chi_A(x) \chi_{\{(x, v) \in \mathbb{S}^1 \times \mathbb{S}^1 : d(\sigma_1(x), v) \leq \alpha\}} \chi_{\sigma_2(B)}(v) \psi_2(v) dx dv.$$

Dividing by α and letting $\alpha \rightarrow 0$, we obtain:

$$I_0(A, B) = \int_{\mathbb{S}^1} \chi_A(x) \chi_{\sigma_2(B)}(\sigma_1(x)) \psi_2(\sigma_1(x)) dx.$$

Inequality (3.12) of the theorem implies the following inequality:

$$(3.13) \quad I_0(A, B) \leq I_0(A^\sharp, B^\sharp),$$

for all subsets A and B of \mathbb{S}^1 .

Now let $B = \mathbb{S}^1$ in the above identity. Then:

$$I_0(A, \mathbb{S}^1) = \int_{\mathbb{S}^1} \chi_A(x) \psi_2(\sigma_1(x)) dx \leq \int_{\mathbb{S}^1} \chi_{A^\sharp}(x) \psi_2(\sigma_1(x)) dx,$$

or equivalently,

$$\int_A \psi_2(\sigma_1(x)) dx \leq \int_{A^\sharp} \psi_2(\sigma_1(x)) dx,$$

for every measurable set $A \subset \mathbb{S}^1$. Since the inequality is true for every measurable set A , we conclude by Lemma 2.2 and Theorem 2.3 that $\psi_2 \circ \sigma_1$ is symmetric (i.e., $\psi_2(\sigma_1(z)) = \psi_2(\sigma_1(\bar{z}))$) and decreasing, which implies that ψ_2 is decreasing on \mathbb{S}_+^1 . Likewise, $\psi_1 \circ \sigma_2$ is symmetric and decreasing on \mathbb{S}_+^1 , implying that ψ_1 is decreasing on \mathbb{S}_+^1 . Thus, σ_1^{-1} and σ_2^{-1} are concave on \mathbb{S}_+^1 and therefore, σ_1 and σ_2 are convex on \mathbb{S}_+^1 .

Next, we denote by $\tau = \sigma_1^{-1} \circ \sigma_2$. With this notation, I_0 becomes:

$$\begin{aligned} I_0(A, B) &= \int_{\mathbb{S}^1} \chi_A(x) \chi_{\sigma_2(B)}(\sigma_1(x)) [\psi_2 \circ \sigma_1](x) dx \\ &= \int_{\mathbb{S}^1} \chi_A(x) \chi_{\tau(B)}(x) [\psi_2 \circ \sigma_1](x) dx = \int_{A \cap \tau(B)} [\psi_2 \circ \sigma_1](x) dx. \end{aligned}$$

We will show that τ is symmetric, i.e., $\tau(\bar{x}) = \overline{\tau(x)}$, for every $x \in \mathbb{S}^1$. Suppose this is not the case. Then there exists $x = e^{i\theta}$, with $\theta \in (0, \pi)$, such that $\overline{\tau(x)} \neq \tau(\bar{x})$. Let $B = \{e^{it} : |t| < \theta\} = B^\sharp$ and $A = \tau(B) \neq A^\sharp$. Then, we have that $A^\sharp \cap \tau(B^\sharp) \subset A \cap \tau(B) = A$ and $|A \cap \tau(B)| > |A^\sharp \cap \tau(B^\sharp)|$. Since $\psi_2 \circ \sigma_1$ is positive, it follows that $I_0(A, B) > I_0(A^\sharp, B^\sharp)$, which contradicts (3.13). Thus, $\sigma_1^{-1} \circ \sigma_2$ is symmetric. We have shown before that $\psi_1 \circ \sigma_2$ is also symmetric.

Claim: $\sigma_1^{-1} \circ \sigma_2$ and $\psi_1 \circ \sigma_2$ symmetric imply σ_2 is symmetric.

Proof of claim: We define f_2 on the interval $[-\pi, \pi]$ as follows:

$$\sigma_2(e^{i\theta}) = e^{if_2(\theta)}.$$

Since $\psi_1 \circ \sigma_2$ is symmetric and $[\psi_1 \circ \sigma_2](e^{i\theta}) = \psi_1(e^{if_2(\theta)}) = \tau_1'(f_2(\theta))$, as in (3.5), it follows that $\tau_1' \circ f_2$ is even.

Since $[\sigma_1^{-1} \circ \sigma_2](e^{i\theta}) = e^{i\tau_1(f_2(\theta))}$ is symmetric, it follows that $\tau_1 \circ f_2$ is odd.

Now, $(\tau_1 \circ f_2)' = (\tau_1' \circ f_2) \cdot f_2'$ is even and $\tau_1' \circ f_2$ is also even (as we have previously shown) and nonzero, so that f_2' is even and thus f_2 is odd. Therefore σ_2 is symmetric and the proof of the claim is now complete.

Following exactly the same steps, we can show that σ_1 is symmetric. We have shown that σ_1, σ_2 are symmetric and convex on \mathbb{S}_+^1 . \square

Corollary 3.3. *With σ , α and $E = \{(x, y) \in \mathbb{S}^1 : d(\sigma(x), y) \leq \alpha\}$, as in Theorem 3.1, we have the following result: For every $f, g : \mathbb{S}^1 \rightarrow \mathbb{R}_+$ positive measurable functions, and every $\alpha > 0$,*

$$(3.14) \quad \int_E f(x)g(y) dx dy \leq \int_E f^\sharp(x)g^\sharp(y) dx dy,$$

if and only if, σ is symmetric, and convex on \mathbb{S}_+^1 .

To sketch the proof, we write f and g as the integrals of their level sets, using the layer-cake representation formula (1.2):

$$\begin{aligned} f(x) &= \int_0^\infty \chi_{\{f>t\}}(x) dt \quad \text{and} \\ g(y) &= \int_0^\infty \chi_{\{g>t\}}(y) dt, \end{aligned}$$

and we notice that $\{f > t\}^\sharp = \{f^\sharp > t\}$ and $\{g > t\}^\sharp = \{g^\sharp > t\}$ so that inequality (3.14) reduces to the case where f and g are characteristic functions, and thus, Theorem 3.1 applies.

Corollary 3.4. *Let σ_1, σ_2 and $E = \{(x, y) \in \mathbb{S}^1 \times \mathbb{S}^1 : d(\sigma_1(x), \sigma_2(y)) \leq \alpha\}$ be as in Theorem 3.2. For every $f, g : \mathbb{S}^1 \rightarrow \mathbb{R}_+$ positive measurable functions, and every $\alpha > 0$,*

$$(3.15) \quad \int_E f(x)g(y) dx dy \leq \int_E f^\sharp(x)g^\sharp(y) dx dy,$$

if and only if, σ_1 and σ_2 are symmetric, and convex on \mathbb{S}_+^1 .

The proof of Corollary 3.4 is indeed very similar to the proof of Corollary 3.3, in which one represents f and g as integrals of the characteristic functions of their level sets.

The next theorem is a generalization of the previous results, where one replaces the product by a function Ψ defined as follows:

$\Psi : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ vanishes on the boundary of \mathbb{R}_+^2 , i.e., $\Psi|_{\{x_1=0\}} = \Psi|_{\{x_2=0\}} = 0$, and

$$\Psi(x_1, x_2) + \Psi(y_1, y_2) \leq \Psi(x_1 \wedge x_2, y_1 \wedge y_2) + \Psi(x_1 \vee x_2, y_1 \vee y_2).$$

If Ψ is twice continuously differentiable, then the above inequality is equivalent to $\partial_{12}\Psi \geq 0$.

Crowe, Zweibel and Rosenbloom [6] noticed that a continuous such Ψ is the distribution function of a Borel measure μ on \mathbb{R}_+^2 , i.e.,

$$(3.16) \quad \Psi(s, t) = \mu([0, s] \times [0, t]),$$

and using Fubini's theorem:

$$(3.17) \quad \int \Psi(f(x), g(y)) dx dy = \int_{\mathbb{R}_+^2} \left[\int \chi_{\{f>s\}}(x) \chi_{\{g>t\}}(y) dx dy \right] d\mu(s, t).$$

We are now ready to state our next result.

Theorem 3.5. *With σ_1 , σ_2 and $E = \{(x, y) \in \mathbb{S}^1 \times \mathbb{S}^1 : d(\sigma_1(x), \sigma_2(y)) \leq \alpha\}$ as in Theorem 3.2, and Ψ the distribution function of a Borel measure μ on \mathbb{R}_+^2 as in (3.16), the following inequality holds for every $\alpha > 0$:*

$$\int_E \Psi(f(x), g(y)) dx dy \leq \int_E \Psi(f^\#(x), g^\#(y)) dx dy,$$

if and only if, σ_1 and σ_2 are symmetric on \mathbb{S}^1 , and convex on \mathbb{S}_+^1 .

Again, we can reduce $\Psi(f(x), g(y))$ to a product of characteristic functions, using (3.17), and the result follows from Theorem 3.2.

The next theorem shows that we can replace the characteristic function of the set E by a decreasing function of the distance between $\sigma_1(x)$ and $\sigma_2(y)$, call it $k[d(\sigma_1(x), \sigma_2(y))]$.

Theorem 3.6. *Let σ_1 , σ_2 be as in Theorem 3.2 and let $k : [0, \infty) \rightarrow [0, \infty)$ be a decreasing function, and Ψ the distribution function of a Borel measure μ on \mathbb{R}_+^2 as in (3.16). Then, the following inequality holds for every decreasing function k ,*

$$\begin{aligned} \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \Psi(f(x), g(y)) k[d(\sigma_1(x), \sigma_2(y))] dx dy \\ \leq \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \Psi(f^\#(x), g^\#(y)) k[d(\sigma_1(x), \sigma_2(y))] dx dy, \end{aligned}$$

if and only if, σ_1 and σ_2 are symmetric on \mathbb{S}^1 , and convex on \mathbb{S}_+^1 .

Proof. Using (1.2), we can write:

$$k(\tau) = \int_0^\infty \chi_{\{k>t\}}(\tau) dt = \int_0^\infty \chi_{[0, l(t)]}(\tau) dt,$$

and substituting $d(\sigma_1(x), \sigma_2(y))$ for τ in the above formula, we have

$$(3.18) \quad k[d(\sigma_1(x), \sigma_2(y))] = \int_0^\infty \chi_{[0, l(t)]}[d(\sigma_1(x), \sigma_2(y))] dt.$$

We define the set $E_{l(t)}$ as follows:

$$E_{l(t)} = \{(x, y) : d(\sigma_1(x), \sigma_2(y)) \leq l(t)\}.$$

Then

$$\chi_{[0, l(t)]}[d(\sigma_1(x), \sigma_2(y))] = 1 \Leftrightarrow (x, y) \in E_{l(t)}.$$

Using this fact, (3.18), Fubini's theorem and Theorem 3.5 we obtain the conclusion of Theorem 3.6 by:

$$\begin{aligned} & \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \Psi(f(x), g(y)) k[d(\sigma_1(x), \sigma_2(y))] dx dy \\ &= \int_0^\infty \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \Psi(f(x), g(y)) \chi_{E_{l(t)}}(x, y) dx dy dt \\ &\leq \int_0^\infty \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \Psi(f^\sharp(x), g^\sharp(y)) \chi_{E_{l(t)}}(x, y) dx dy dt \\ &= \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \Psi(f^\sharp(x), g^\sharp(y)) k[d(\sigma_1(x), \sigma_2(y))] dx dy. \end{aligned}$$

□

REFERENCES

- [1] A. Baernstein, II. Convolution and rearrangement on the circle. *Complex Variables Theory Appl.*, 12(1-4):33–37, 1989.
- [2] A. Baernstein, II and B. A. Taylor. Spherical rearrangements, subharmonic functions, and $*$ -functions in n -space. *Duke Math. J.*, 43(2):245–268, 1976.
- [3] A. Baernstein II. A unified approach to symmetrization. *Partial Differential Equations of Elliptic Type*, eds. A. Alvino et al, *Symposia Mathematica*, Cambridge Univ. Press, 35:47–91, 1995.
- [4] C. Bennett and R. Sharpley. *Interpolation of operators*. Academic Press Inc., 1988.
- [5] H. J. Brascamp, E. H. Lieb, and J. M. Luttinger. A general rearrangement inequality for multiple integrals. *J. Funct. Anal.*, 17:227–237, 1974.
- [6] J. A. Crowe, J. A. Zweibel, and P. C. Rosenbloom. Rearrangements of functions. *J. Funct. Anal.*, 66(3):432–438, 1986.
- [7] H. Federer. *Geometric measure theory*. Springer-Verlag New York Inc., New York, 1969.
- [8] H. Hadwiger and D. Ohmann. Brunn-Minkowskischer Satz und Isoperimetrie. *Math. Z.*, 66:1–8, 1956.
- [9] G. H. Hardy, J. E. Littlewood, and G. Pólya. *Inequalities*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1988. Reprint of the 1952 edition.
- [10] E. H. Lieb and M. Loss. *Analysis*, volume 14 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2001.
- [11] R. Osserman. The isoperimetric inequality. *Bull. Amer. Math. Soc.*, 84(6):1182–1238, 1978.
- [12] F. Riesz. Sur une inégalité intégrale. *J. London Math. Soc.*, 5:162–168, 1930.
- [13] S. L. Sobolev. On a theorem of functional analysis. *Mat. Sb. (N.S.)*, 4:471–497, 1938. English transl., *Amer. Math. Soc. Transl.*, 2(34):29–68, 1963.

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